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Reduction to quadratures of integrable generalisations of the Calogero system

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Abstract. It is shown here that the Calogero system of three particles of equal or different masses, interacting in one-dimension via an arbitrary translationally invariant homogeneous potential of order −2 and confined by an external potential, is separable. An explicit solution is given for a case distinguished by the completely degenerate character of the motion.

The classical Calogero system (Calogero 1969) of three interacting particles with equal masses and confined in a one-dimensional harmonic well is described by the Hamiltonian

$$H_c = (1/2m)(p_1^2+p_2^2+p_3^2) + g^2[(x_1-x_2)^{-2}+(x_2-x_3)^{-2}+(x_3-x_1)^{-2}] + \tfrac{1}{2}\omega^2(x_1^2+x_2^2+x_3^2). \tag{1}$$

This system is completely integrable and can be solved explicitly (Khandekar and Lawande 1972) in terms of circular functions through the separation of variables. The motion is strictly periodic since the Calogero system, as a completely degenerate system, has a total of five independent, integrals of motion that do not depend explictly on time.

The integrability of the Calogero system is not only limited to the case of equal masses. For particles of different masses, it has been reduced to quadratures by Jacobi (1886). Both cases belong to a considerably larger class of integrable systems (Wojciechowski 1983) characterised by the Hamiltonians

$$H_1 = T + V_{-2} + g(I_1) + h(X) \tag{2}$$

$$H_2 = T + V_{-2} + f(I) \tag{3}$$

where

$$T = (p_1^2+p_2^2+p_3^2)/2m, \qquad X = x_1+x_2+x_3,$$

$$I = \tfrac{1}{2}(x_1^2+x_2^2+x_3^2), \qquad I_1 = \tfrac{1}{6}[(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2]$$

and by

$$H_3 = T' + V_{-2} + g(I_1') + h(X') \tag{4}$$

$$H_4 = T' + V_{-2} + f(I') \tag{5}$$

where

$$T' = p_1^2/2m_1 + p_2^2/2m_2 + p_3^2/2m_3, \qquad X' = m_1x_1 + m_2x_2 + m_3x_3$$

$$I' = \tfrac{1}{2}(m_1x_1^2 + m_2x_2^2 + m_3x_3^2),$$

$$I_1' = [m_1m_2(x_1 - x_2)^2 + m_2m_3(x_2 - x_3)^2 + m_3m_1(x_3 - x_1)^2]/2M,$$

$$M = m_1 + m_2 + m_3.$$

The potential V_{-2} is here an arbitrary translationally invariant homogeneous function of order -2. The mass dependence of the Hamiltonians (4) and (5) is non-trivial since the simple canonical change of variables $p_k' = p_k m_k^{-1/2}$, $x_k' = m_k^{1/2} x_k$ destroys the translational invariance of V_{-2}. Note also that usually the natural Hamiltonian systems are integrable for very particular values of the masses (see e.g. Bountis *et al* 1982) while here the masses are the parameters of the system.

The sets of three independent commuting integrals of motion for systems (2)–(5) have been constructed by the group theoretic method (Wojciechowski 1983) but the way of solving them seems to be unknown.

The particular feature of the integrals given by Wojciechowski is their quadratic (in momentum) character which strongly suggests the possibility of separating variables. Another hint in favour of the separability of the Hamiltonians (2)–(5) is their dependence on arbitrary functions which happens, in practice, only for separable systems. However, the way of finding the adequate variables is not a trivial task.

The aim of this paper is to construct the separation variables for all systems (2)–(5) in order to reduce them to quadratures. To do this the general form of the potential V_{-2} has to be found first. From the condition of translational invariance

$$\partial V_{-2}/\partial x_1 + \partial V_{-2}/\partial x_2 + \partial V_{-2}/\partial x_3 = 0$$

we have $V_{-2} = \Phi[(x_1 - x_2), (x_2 - x_3)]$, where Φ is an arbitrary differentiable function of variables $y_1 = x_1 - x_2$ and $y_2 = x_2 - x_3$. Further, by the homogeneity of the order -2, the potential V_{-2} has also to satisfy the equation

$$x_1\, \partial V_{-2}/\partial x_1 + x_2\, \partial V_{-2}/\partial x_2 + x_3\, \partial V_{-2}/\partial x_3 = -2V_{-2}$$

which in the variables y_1, y_2 reads

$$y_1\, \partial\Phi/\partial y_1 + y_2\, \partial\Phi/\partial y_2 = -2\Phi$$

and has as the general solution $\Phi(y_1, y_2) = y_2^{-2}\Psi(y_1/y_2)$ where Ψ is an arbitrary differentiable function of one variable. Thus we obtain $V_{-2} = (x_2 - x_3)^{-2}\Psi((x_1 - x_2)/(x_2 - x_3))$ and in the 'cylindrical' coordinates R, r, φ defined by

$$R = \tfrac{1}{3}(x_1 + x_2 + x_3), \qquad x = 2^{-1/2}(x_1 - x_2), \qquad y = 6^{-1/2}(x_1 + x_2 - 2x_3) \tag{6}$$

$$x = r\sin\varphi, \qquad y = r\cos\varphi \tag{7}$$

we have

$$V_{-2} = r^{-2} 2 \sin^{-2}(\tfrac{2}{3}\pi + \varphi)\Psi(-\sin\varphi/\sin(\tfrac{2}{3}\pi + \varphi)).$$

The variable R is the coordinate of the centre of mass; $r^2 = \frac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2]$ measures the mean quadratic length, while φ is the phase of relative configuration of particles. In these variables the Hamiltonian (2) reads

$$H_2 = \tfrac{1}{6}p_R^2 + \tfrac{1}{2}p_r^2 + (1/2r^2)p_\varphi^2 + (1/r^2)F(\varphi) + g(\tfrac{1}{2}r^2) + h(3R)$$

and by the standard method of separation of the Hamilton–Jacobi equation (Landau and Lifshitz 1960) we get three commuting functionally independent integrals of motion

$$\tfrac{1}{6}p_R^2+h(3R)=E_R, \qquad \tfrac{1}{2}p_\varphi^2+F(\varphi)=E_\varphi, \qquad \tfrac{1}{2}p_r^2+g(\tfrac{1}{2}r^2)+(1/r^2)E_\varphi=E_r \tag{8}$$

So from the relations $3m\dot{R}=p_R$, $m\dot{r}=p_r$, $mr^2\dot{\varphi}=p_\varphi$ we can immediately express in quadratures the quantities $R(t)$, $r(t)$, and $\varphi(t)$. Similarly, by introducting the 'spherical' coordinates ρ, θ, φ defined by

$$3^{1/2}R=\rho\cos\theta, \qquad r=\rho\sin\theta, \qquad \varphi=\varphi$$

(where $\rho^2=x_1^2+x_2^2+x_3^2$) the Hamiltonian (3) can also be reduced to the separable form

$$H_3=\tfrac{1}{2}p_\rho^2+(1/2\rho^2)p_\theta^2+(1/2\rho^2\sin^2\theta)p_\varphi^2+(1/\rho^2\sin^2\theta)F(\varphi)+f(\tfrac{1}{2}\rho^2).$$

This has the three following independent commuting integrals of motion

$$\begin{aligned}&\tfrac{1}{2}p_\rho^2+F(\varphi)=E_\varphi, \qquad \tfrac{1}{2}p_\theta^2+(1/\sin^2\theta)E_\varphi=E_\theta,\\ &\tfrac{1}{2}p_\rho^2+(1/\rho^2)E_\theta+f(\tfrac{1}{2}\rho^2)=E_\rho.\end{aligned} \tag{9}$$

To find the proper separation variables for systems (4) and (5), it is convenient to consider first the Calogero system with different masses and different coupling constants g_k:

$$H_d=\frac{1}{2}\left(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}+\frac{p_3^2}{m_3}\right)+\left[\frac{g_1}{(x_1-x_2)^2}+\frac{g_2}{(x_2-x_3)^2}+\frac{g_3}{(x_3-x_1)^2}\right]+\frac{1}{2}\omega^2(m_1x_1^2+m_2x_2^2+m_3x_3^2) \tag{10}$$

which is also interesting in itself. As one of the variables we take again the coordinate of the centre of mass

$$R=M^{-1}(m_1x_1+m_2x_2+m_3x_3) \tag{11}$$

and, generalising formulae (6), we assume

$$x_2-x_3=\alpha y-(\beta/m_2)x \qquad x_1-x_3=\alpha y+(\beta/m_1)x \qquad x_1-x_2=(x_1-x_3)-(x_2-x_3) \tag{12}$$

where α and β are to be determined. We can find them by requiring $m_1x_1^2+m_2x_2^2+m_3x_3^2=x^2+y^2+MR^2$ as a natural generalisation of the identity $x_1^2+x_2^2+x_3^2=x^2+y^2+3R^2$ satisfied by the coordinates (6). Then employing the identity

$$\begin{aligned}&m_1m_2(x_1-x_2)^2+m_2m_3(x_2-x_3)^2+m_3m_1(x_3-x_1)^2\\ &\quad=M(m_1x_1^2+m_2x_2^2+m_3x_3^2)-(m_1x_1+m_2x_2+m_3x_3)^2\end{aligned}$$

we obtain $\alpha=[M/m_3(m_1+m_2)]^{1/2}$, $\beta=[m_1m_2/(m_1+m_2)]^{1/2}$. In the modified 'cylindrical' coordinates R, r, φ, defined by formulae (11) and (12) and by the relations

$$x=r\sin\varphi, \qquad y=r\cos\varphi, \tag{13}$$

the Hamiltonian (10) reads

$$\begin{aligned}H_d=&\tfrac{1}{2}p_r^2+\frac{1}{2r^2}p_\varphi^2+\frac{1}{2M}p_R^2\\ &+\frac{1}{r^2}\left(\frac{g_1}{\{[(m_1+m_2)/m_1m_2]\beta\sin\varphi\}^2}+\frac{g_2}{[\alpha\cos\varphi-(\beta/m_2)\sin\varphi]^2}\right.\\ &\left.+\frac{g_3}{[\alpha\cos\varphi+(\beta/m_1)\sin\varphi]^2}\right)+\tfrac{1}{2}\omega^2r^2+\tfrac{1}{2}\omega^2MR^2\end{aligned} \tag{14}$$

and the related integrals have the form

$$(1/2M)p_R^2 + \tfrac{1}{2}\omega^2 M R^2 = E_R, \qquad \tfrac{1}{2}p_\varphi^2 + F'(\varphi) = E_\varphi,$$
$$\tfrac{1}{2}p_r^2 + \tfrac{1}{2}\omega^2 r^2 + (1/r^2)E_\varphi = E_r \tag{15}$$

where $F'(\varphi)$ denotes the term in the Hamiltonian (4) which depends only on φ. It is easy to see that the Hamiltonian (14) separates also in the modified spherical coordinates defined by formulae (11), (12), (13) and by relations $M^{1/2}R = \rho\cos\theta$, $r = \rho\sin\varphi$. Then

$$H_d = \tfrac{1}{2}p_\rho^2 + (1/2\rho^2)p_\theta^2 + (1/2\rho^2\sin^2\theta)p_\varphi^2 + (1/\rho^2\sin^2\theta)F'(\varphi) + \tfrac{1}{2}\omega^2\rho^2.$$

Only one of the integrals

$$\tfrac{1}{2}p_\varphi^2 + F'(\varphi) = E_\varphi, \qquad \tfrac{1}{2}p_\theta^2 + (1/\sin^2\theta)E_\varphi = E_\theta, \qquad \tfrac{1}{2}p_\rho^2 + \frac{1}{\rho^2}E_\theta + \tfrac{1}{2}\omega^2\rho^2 = E_\rho$$

is independent of the integrals (15) because $E_\rho = E_r + E_R$. So system (10) has four functionally independent integrals not depending explicitly on time (E_R, E_r, E_φ, E_θ), while system (1) has five. Nevertheless, it is suprising that the Calogero system with different masses is still partially degenerate since the arbitrariness of masses usually destroys integrability.

Having introduced the modified cylindrical coordinates, we can easily see that the Hamiltonian (4) has three integrals given by formulae (8) with a minor difference that h depends on MR instead of $3R$. Analogously the Hamiltonian (5) separates in the modified 'spherical' coordinates and its integrals are given by (9).

In all cases (2)–(5) considered above, the motion of the system can be expressed in quadratures but the integrals to be performed are of an elliptic or more general type. However, for a particular potential

$$\begin{aligned} V_{-2}(x_1, x_2, x_3) = {} & g^2[(x_1-x_2)^{-2} + (x_2-x_3)^{-2} + (x_3-x_1)^{-2}] \\ & + f^2[(x_1+x_2-2x_3)^{-2} + (x_2+x_3-2x_1)^{-2} + (x_3+x_1-2x_2)^{-2}] \\ & + \tfrac{1}{2}\omega^2(x_1^2+x_2^2+x_3^2) \end{aligned} \tag{16}$$

which generalises the Calogero potential in (1) all integrations can be performed explicitly in terms of circular functions and the motion is strictly periodic. This has not been difficult to presume since the spectrum of the corresponding quantum problem is equidistant (Wolfes 1974) as happens only in the systems with the highest possible symmetry. So we consider here system (1) with potential (16) for the case $g^2 > 0$, $f^2 > 0$ and $\omega^2 > 0$ to exclude the collapse of the particles and we adopt the ordering of particles $x_1 > x_2 > x_3$ which is preserved in time. The mass of the particles can be taken as equal to unity since momenta and the parameters f^2, g^2, ω^2 can always be properly rescaled. This system has two equilibrium configurations corresponding in cylindrical coordinates (6), (7) to $R = 0$, $r = [(9g + 3^{1/2}f)/3\omega]^{1/2}$, $\cot 3\varphi_{1,2} = \pm[f/g \times 27^{1/2}]^{1/2}$. These are asymmetric with respect to reflections in the origin because for $\varphi \to \tfrac{1}{2}\pi$ (i.e. for $x_2 \to \tfrac{1}{2}(x_1+x_3)$) the energy of the system goes to infinity and, therefore, the configuration in which the central particle is equidistant from two extreme particles is forbidden. So the physical motion can take place in two separate regions around the equilibrium points and the phase space splits into two simply connected components.

The Hamiltonian of the system, expressed in 'cylindrical' coordinates, has the form

$$H_w = \tfrac{1}{6}p_R^2 + \tfrac{1}{2}p_r^2 + (1/2r^2)p_\varphi^2 + \tfrac{3}{2}\omega^2 R^2 + \tfrac{1}{2}\omega^2 r^2 + \tfrac{1}{2}9g^2(1/r^2\sin^2 3\varphi) + \tfrac{1}{6}f^2(1/r^2\cos^2 3\varphi) \tag{17}$$

and the corresponding integrals are

$$\tfrac{1}{6}p_R^2 + \tfrac{3}{2}\omega^2 R^2 = E_R, \qquad \tfrac{1}{2}p_\varphi^2 + \tfrac{1}{2}9g^2(1/\sin^2 3\varphi) + \tfrac{1}{6}f^2(1/\cos^2 3\varphi) = E_\varphi,$$

$$\tfrac{1}{2}p_r^2 + \tfrac{1}{2}\omega^2 r^2 + (1/r^2)E_\varphi = E_r$$

From these integrals we find the equations of motion

$$\mathrm{d}R/\mathrm{d}t = (\tfrac{2}{3}E_R - \omega^2 R^2)^{1/2}, \qquad \mathrm{d}r/\mathrm{d}t = (2E_r - \omega^2 r^2 - 2E_\varphi/r^2)^{1/2},$$

$$\mathrm{d}\varphi/\mathrm{d}t = (1/r^2)(2E_\varphi - 9g^2/\sin^2 3\varphi - f^2/3\cos^2 3\varphi)^{1/2}$$

and finally

$$R(t) = (\tfrac{2}{3}E_R)^{1/2}(1/\omega)\sin(\omega t + C_1)$$

$$r(t) = (1/\omega)[(E_r^2 - 2E_\varphi\omega^2)^{1/2}\sin(2\omega t + C_2) + E_r]^{1/2}$$

$$\cos^3 3\varphi(t) = \frac{1}{2A}(B^2 - 4Af^2)^{1/2}\sin\left\{6\tan^{-1}\left[\frac{E_r}{(2E_\varphi)^{1/2}\omega}\tan(\omega t + C_2')\right.\right.$$

$$\left.\left. + \left(\frac{E_r^2}{2\omega^2 E_\varphi} - 1\right)^{1/2}\right] + C_3'\right\} + \frac{B}{2A}$$

where $A = 6E_\varphi$, $B = 6E_\varphi + f^2 - 27g^2$ and C_1, C_2, C_1', C_2' are the constants of integration. The action-angle variables, defined as the integrals $I_q = 1/2\pi \oint p_q\, \mathrm{d}q$ over one cycle of the motion of variable q, have here the form

$$I_R = E_R/\omega, \qquad I_r = E_r/2\omega - \tfrac{1}{2}(2E_\varphi)^{1/2}, \qquad I_\varphi = \tfrac{1}{6}(2E_\varphi)^{1/2} - \tfrac{1}{2}g - f/6\times 3^{1/2}.$$

Hamiltonian (17) expressed in these variables reads

$$H_\omega = \omega I_R + 2\omega I_r + 6\omega I_\varphi + 3g\omega + 3^{-1/2}f\omega. \tag{18}$$

It is worth noting that the basic frequency $\omega_\varphi = \partial H_\omega/\partial I_\varphi$ related to the variable φ is here twice as great as that for the Calogero system (1). This reflects the split of the phase space into two separate components. The additive term in the Hamiltonian (18) is equal to the minimal energy of the system in either of the equilibrium configurations.

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